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On an exactly solvable compact cluster growth model on a square lattice

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Abstract

We consider a cluster growth model on a square lattice that is in the compact directed percolation universality class. The model is exactly solvable as regards its static cluster properties, and expressions are given for various quantities of interest, such as the mean perimeter length and the mean cluster size. The results also provide new information on area–perimeter generating functions for a class of self-avoiding polygons on square lattices.

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1. Introduction

Stochastic growth models on lattices have been widely studied, both with regard to their static properties (i.e. characterizing clusters) and also their dynamic properties. Epidemics, forest fires, avalanches and aggregates have all been studied on this basis (see, e.g., [1,2]). Two well known examples are the Eden model [1,2], which has played an important role in modelling certain biological processes and interface roughening, and the Leath model [1–3], which has greatly aided our understanding of isotropic percolation. However, although much is known about the properties of such models, most have not yet been solved exactly except in one spatial dimension.

In this paper we consider a different stochastic growth model on a square lattice which generates 'spherical' clusters in discrete time. The clusters are compact (similar to the Eden model, but in a stricter sense) and the growth process can terminate (similar to the Leath model). As regards its static properties, the model is exactly solvable and belongs to the compact directed percolation (CDP) universality class (in $(1+1)$ dimensions) [4,5]. Directed percolation models in general are widely studied as examples of systems with an absorbing state (i.e. non-equilibrium systems) that exhibit a phase transition [6]. We are interested in characterizing the clusters according to their perimeter and area.

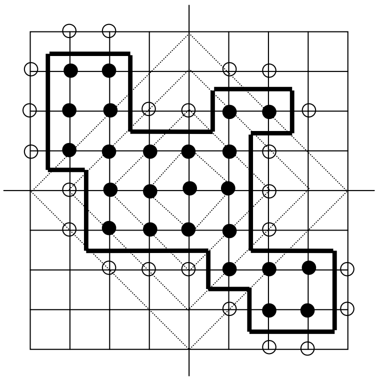

Figure 1. A cluster generated in six time steps by the CCG model. The dotted lines link sites on a shell. The full circles are occupied sites and the open circles are the unoccupied sites that contribute a factor q to the diagram weighting. The full curve is the perimeter polygon. For this cluster, $\ell = 32$, $s = 24$ and the weighting is $p^{14}q^{23}$.

Besides its virtue of being exactly solvable, the model studied is also closely related to various combinatorial enumeration problems (in particular, classes of self-avoiding polygons). The problem of enumerating all self-avoiding polygons has remained unsolved [7], although much is known from series expansions (see, e.g., [8]). For so-called staircase polygons, the area–perimeter generating functions have been determined exactly [7, 9], and the work presented here provides a different perspective on these results. In addition, we can provide exact generating functions for a different class of self-avoiding polygons. With an eye to the future, it has been noted that the area–perimeter generating function for staircase polygons obeys a non-linear functional equation [10], from which the properties of various scaling functions can be derived [11]. This is also found to be true of other restricted classes of self-avoiding polygons [10, 12]. Very recent developments [13, 14] suggest that this concept may be of much wider applicability and may represent a profound new approach to studying longstanding problems encountered in other lattice models. In this context, the present work is also of some interest, in that an equivalent functional structure may well exist, but has yet to be identified.

2. The compact cluster growth model

The compact cluster growth (CCG) model is specified as follows (see figure 1). Sites on a square lattice are labelled by integer pairs (x, y). At time $t = 0$, the origin is occupied with probability 1 and all other sites are empty. At discrete time t, the occupancy of every site in the shell whose chemical distance from the origin is $|x| + |y| = t$ is determined in parallel, depending upon the occupancy of the nearest-neighbour sites in the preceding shell, whose chemical distance from the origin is $t - 1$. If $x \neq 0$ and $y \neq 0$ then there are two such neighbouring sites and the conditional probability rules are as follows: $P[1|0, 0] = 0$, $P[1|1, 0] = P[1|0, 1] = p$ and $P[1|1, 1] = 1$, where p thresholds a random number uniformly chosen on the interval [0, 1], and a site value of 1 (0) corresponds to occupied (unoccupied). For the four sites on the axes which have either $x = 0$ or $y = 0$ there is only one neighbouring site on the preceding shell. For these sites the rule is simply: $P[1|0] = 0$ and $P[1|1] = p$. In this way, the occupancy of successive shells is updated in time and the clusters so generated are fully compact (ensured by the condition $P[1|1, 1] = 1$). The cluster growth terminates if

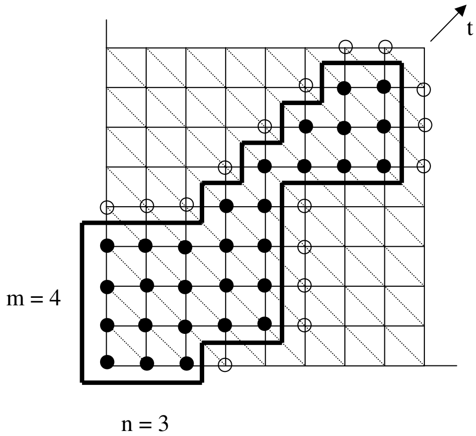

Figure 2. A typical cluster for CDP in (1 + 1) dimensions generated in 14 time steps, with $m = 4$ and $n = 3$. For this cluster, $\ell = 32$, $s = 29$ and the weighting is $p^{14}q^{16}$.

$p < p_c$, and may or may not terminate if $p > p_c$, where $p_c = 1/2$ (see below). An example of a finite cluster generated by this process is shown in figure 1.

Such clusters can be characterized by their perimeter length and area, and by their probability of occurrence. The perimeter length, ℓ, is defined to be the length of the perimeter of the self-avoiding polygon (which is defined on the dual lattice) that bounds the compact cluster as tightly as possible (see figure 1). The area is simply the number of occupied sites in the cluster. The probabilistic weighting of a given cluster follows directly from the growth rules, in which the condition $P[1|1, 1] = 1$ plays a prominent role. Thus, only sites that have *one* occupied nearest neighbour on the preceding 'chemical shell' contribute a factor of p to the cluster weighting (e.g. in figure 1 there are 14 of these). All the other occupied sites are occupied with probability 1 (including the origin) and so contribute nothing to the weighting (in figure 1 there are 10 of these). All empty sites that were otherwise *accessible* during the cluster growth contribute a factor of $q = 1 - p$ to the cluster weighting (e.g. in figure 1 there are 23 of these). It turns out that there is no simple way to express the weight of a given CCG cluster simply in terms of its perimeter length or area. Nevertheless, it is possible, through the use of suitable generating functions, to evaluate quantities such as the mean perimeter length and the mean cluster size.

There is a close link between the CCG model and CDP in (1 + 1) dimensions. The latter is simple to define (the nomenclature 'CDP' is standard although CDP is not actually in the directed percolation universality class [6]). Sites on a one-dimensional lattice $(s_i(t))$ are updated in parallel on the basis of defined conditional transition probabilities $P[s_i(t + 1)|s_{i-1}(t), s_{i+1}(t)]$, namely $P[1|0, 0] = 0$, $P[1|1, 0] = P[1|0, 1] = p$ and $P[1|1, 1] = 1$, where p thresholds a random number uniformly chosen on [0, 1]. The usual initial condition is that a single site (the origin) is occupied. Representing successive states of the lattice $\{s_i(t)\}$ on a space–time diagram generates two-dimensional CDP clusters on a (diagonal) square lattice which are compact (see figure 2). As first noted in [4], and developed in [5], this means that the (finite) clusters may be placed in one-to-one correspondence with pairs of directed parallel walks on the dual lattice which intersect only on the first and last step (see figure 2). Since the enumeration of such walks is well understood [15–17], various quantities of interest can be calculated exactly. For example, the critical percolation probability $p_c = 1/2$ [4], and the principal critical exponents are all known [5]. There have also been further exact results given for CDP in the presence of a physical wall [18–22].

The directed parallel walks define a perimeter polygon in the sense given above, and one can refer to the perimeter length and area (size) of CDP clusters in the same way as for CCG clusters. Following [5] we note that each walk consists of $\ell/2$ steps. Each 'outward' step of a given walk (except the first) is associated with a factor of p, whilst each 'inward' step of a given walk is associated with a factor of q (see figure 2). Further, since both walks intersect only at the first and last steps, the number of 'outward' steps on one walk is equal to the number of 'inward' steps on the other, and vice versa. It follows that the perimeter length, ℓ, uniquely determines the weighting of a given CDP cluster, i.e. $p^{-2}(pq)^{\ell/2}$ [5]. Evidently a similar argument does not apply to the more complicated clusters generated by the CCG model.

The perimeter boundary of a CDP cluster is known as a 'staircase' polygon, since the perimeter polygon can be represented by two directed walks (see above) and each of these walks resembles a 'staircase' [7]. It will become clear (see the discussion later in the paper) that the CCG clusters (e.g. figure 1) are closely related to staircase polygons through concatenation. The subtle feature is that the four such staircase polygons required (one for each quadrant) are correlated through having common sites along the axes; handling this correlation exactly is the main theme of this paper. Since the growth of axis sites is essentially a one-dimensional process (determined only by the state of the preceding site on the same axis), their growth will eventually terminate. Only after such a point is reached, however, will the 'lobes' of the growing clusters decouple. We note in passing that the clusters studied here are quite distinct from the directed compact clusters (lattice animals) studied in [23, 24].

3. Area–perimeter generating functions

3.1. CDP in (1 + 1) dimensions

We begin by summarizing some well-known results for CDP clusters in (1 + 1) dimensions which will be useful in what follows. The perimeter generating function of the bounding staircase polygons is easily obtained from the properties of random walks [5, 15–17]:

$$G(y) \equiv \sum_{\ell} C_\ell y^\ell = \tfrac{1}{2}[1 - 2y^2 - \sqrt{1-4y^2}] \tag{1}$$

where y is the perimeter 'activity' and C_ℓ is the number of distinct clusters with perimeter length ℓ. The structure of (1) is simple and algebraic. Since the weight of a given CDP cluster is simply $p^{-2}(pq)^{\ell/2}$ (see the previous section), the probability that a given cluster is finite follows immediately:

$$Q \equiv p^{-2} \sum_{\ell} C_\ell (pq)^{\ell/2} = p^{-2} G\big(\sqrt{pq}\big). \tag{2}$$

An important feature of (1) is that $\sqrt{1-4pq} = |1-2p|$. Thus for $p < p_c$ we have $Q_< = 1$ and for $p > p_c$ we have $Q_> = (q/p)^2$, where $p_c = 1/2$. The probability of generating an infinite cluster $P_\infty = 1 - Q$, and so for $p > p_c$ we have $P_\infty = (2p-1)/p^2$ with critical exponent $\beta = 1$. The mean perimeter length for $p < p_c$ is given by

$$L_< = p^{-2}\left(y\frac{\partial G}{\partial y}\right)\bigg|_{y=\sqrt{pq}} = 4\left(\frac{1-p}{1-2p}\right) \tag{3}$$

and this diverges as $p \to p_c$ with critical exponent $\tau = 1$. For $p > p_c$, the corresponding quantity of interest is the mean perimeter length *given* that the clusters are finite, i.e.

$$L_> = Q_>^{-1} p^{-2}\left(y\frac{\partial G}{\partial y}\right)\bigg|_{y=\sqrt{pq}} = 4\left(\frac{1-q}{1-2q}\right). \tag{4}$$

Examination of (3) and (4) reveals a duality under the transformation $p \Leftrightarrow q$, a property that was explained in [5]. It is of interest here since a similar duality in the CCG model turns out to be missing.

To calculate the mean CDP cluster size, S, one needs information about the staircase polygon area–perimeter generating function $G(y, z) \equiv \Sigma_{\ell,s} C_{\ell s} y^\ell z^s$ (where z is the area 'activity' and s is the cluster size). The function $G(y, z)$ itself is much more difficult to obtain than (1) [9, 10]. Further, the resulting non-algebraic expression (involving 'q series') is very unwieldy; even taking the limit $z \to 1^-$ to recover (1) is highly non-trivial [11]. Fortunately, to generate the moments of interest it is not necessary to evaluate $G(y, z)$ for all values of z. One such method was demonstrated in [5], but the most elegant derivation follows from a result due to Prellberg and Brak [10], who showed that the generating function $G(y, z) \equiv F(y^2, y^2, z)$, where $F(x, y, z)$ obeys a non-linear functional equation

$$F(x, y, z) = [F(zx, y, z) + zx][F(x, y, z) + y]. \tag{5}$$

By setting $z = 1$ in (5) one can recover (1) immediately. Further, the mean cluster size can be calculated from (5). For $p < p_c$ one has, after some straightforward algebra,

$$S_< = p^{-2}\left(\frac{\partial F(x, y, z)}{\partial z}\right)\Bigg|_{\substack{x=y=pq\\ z=1}} = \left(\frac{1-p}{1-2p}\right)^2. \tag{6}$$

The critical exponent is therefore $\gamma = 2$. For $p > p_c$ one can, as above, similarly define a mean cluster size given that the clusters are finite:

$$S_> = Q_>^{-1} p^{-2}\left(\frac{\partial F(x, y, z)}{\partial z}\right)\Bigg|_{\substack{x=y=pq\\ z=1}} = \left(\frac{1-q}{1-2q}\right)^2. \tag{7}$$

Again the duality structure is clear. That (6) and (7) follow from (5) is elementary, although the link does not seem to have been written down so explicitly before.

3.2. The present model

The first step to obtaining the area–perimeter generating function for the CCG clusters depicted in figure 1 is to obtain the area–perimeter generating function $g_{mn}(y, z)$ for staircase polygons defined for given values m and n of the number of occupied sites along the two axes (see figure 2). When suitably concatenated, one can then sum over the indices and this will keep track of the correlations (see the later discussion). The function $g_{mn}(y, z)$ has a well-defined recursive structure which is evident from the diagrammatic expansions shown in figure 3 (a 'Temperley' method [7, 9]), and clearly one has the symmetry $g_{mn}(y, z) = g_{nm}(y, z)$. In general one finds that

$$g_{mn} = y^2 z^n [g_{m-1,n} + g_{m-1,n+1} + g_{m-1,n+2} + \cdots]$$

whereupon one can obtain the following recursion relation:

$$z g_{m+1,n} = y^2 z^{n+1} g_{m,n} + g_{m+1,n+1} \qquad m, n \geqslant 1. \tag{8}$$

We have not solved this recursion for general z, but it seems likely that one could do so using 'q-series' techniques similar to those used in [9, 10]. For example, one can sum over m to derive a recursion for $h_n(y, z) \equiv \sum_{m=1}^{\infty} g_{mn}(y, z)$:

$$h_{n+2} - z(1 + y^2 - y^2 z^{n+1})h_{n+1} + y^2 z^2 h_n = 0$$

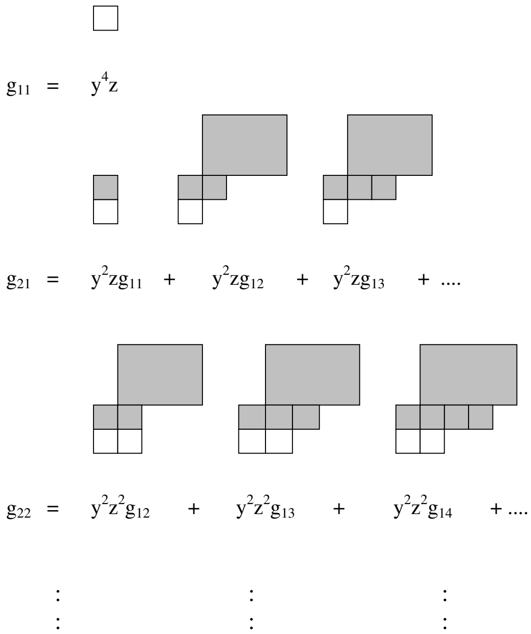

Figure 3. Graphical representations illustrating the recursive structure of the area–perimeter generating functions $g_{mn}(y, z)$.

and this is solved in [9, 10]. However, as mentioned above, one does not need to solve for general z to evaluate many quantities of interest; indeed, it is easier not to do so. When $z = 1$, it is straightforward to show from (8) that

$$g_{mn}(y) = y^4(\lambda_-)^{m+n-2}$$
$$\lambda_- = \frac{1 - \sqrt{1 - 4y^2}}{2} \tag{9}$$

where λ_- is a root of $\lambda^2 - \lambda + y^2 = 0$. In choosing this root one notes that, as the 'activities' y, $z \to 0$, so $g_{mn} \sim z^{mn}y^{2m+2n}$, i.e. the 'ground-state' cluster corresponds to a rectangle of dimensions $m \times n$. Summing (9) over m and n one obtains for all staircase polygons $G(y) = (\lambda_-)^2$, which agrees with (1). Further, (9) plays an important role in determining the probability of generating a finite cluster, and the mean cluster perimeter length, for the compact clusters illustrated in figure 1 (see the next section).

To calculate the mean cluster size one needs additional information about the derivative of g_{mn} with respect to z, evaluated at $z = 1$. This is harder to obtain. Differentiating (8) gives

$$f_{m+1,n} - y^2 f_{mn} - f_{m+1,n+1} = (n+1)y^6(\lambda_-)^{m+n-2} - y^4(\lambda_-)^{m+n-1} \tag{10}$$

where $f_{mn} \equiv \partial g_{mn}/\partial z|_{z=1}$. In finding a solution of this inhomogeneous recursion one uses the fact that, as $y \to 0$, $f_{mn} \sim mny^{2m+2n}$ (see above). The solution of (10) is

$$f_{mn} = \left[\frac{y^4}{2(1-2\lambda_-)}(\lambda_-(1-\lambda_-)(m^2+n^2) - \lambda_-(m+n) + 2(1-\lambda_-)^2 mn) + \frac{y^4\lambda_-^2}{2(1-2\lambda_-)^2}(m+n-2)\right](\lambda_-)^{m+n-2}. \tag{11}$$

In the next section we show how to calculate the mean cluster size using this result.

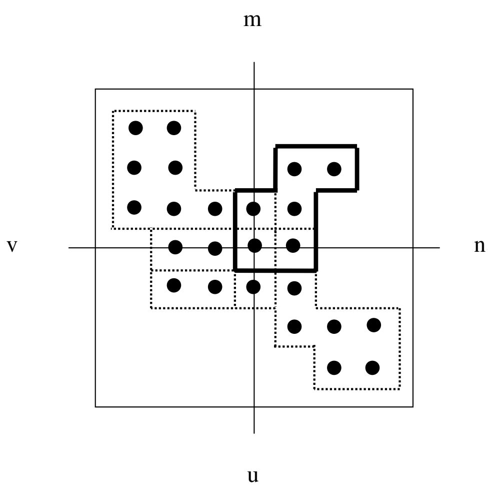

Figure 4. Showing how the cluster in figure 1 arises through the concatenation of four overlapping staircase polygons (one of which is emphasized in bold to aid the eye in following the appropriate lines). The origin site is common to all four polygons; all other axis sites are common to two polygons.

The area–perimeter generating function, $G(y, z)$, for CCG clusters of the type depicted in figure 1 can now be constructed as follows. Every cluster can be thought of as arising from four overlapping staircase polygons that share common sites along the axes (see, e.g., figure 4). Clearly then $G(y, z)$ will be derived from a product of the form $g_{mn} \times g_{nu} \times g_{uv} \times g_{vm}$, suitably corrected for 'over-counting' (thus figure 4 illustrates one of the clusters 'generated' by the product $g_{22}g_{22}g_{23}g_{32}$). By 'over-counting' we mean that, as things stand, the product $g_{mn}g_{nu}g_{uv}g_{vm}$ counts the origin four times, counts all other sites on the axes twice, and counts four unit sections of the CCG cluster perimeter twice. Further, it also counts other sections of the perimeter of the individual staircase polygons that are now *internal* to the CCG cluster polygon and therefore not to be included (see figure 4). This 'over-counting' has to be factored out. A little thought convinces one (the formal proof is omitted here but can be demonstrated on the basis of a term-by-term diagrammatic expansion) that

$$G(y, z) = \frac{z}{y^4} \sum_{m,n,u,v=1}^{\infty} \frac{g_{mn}g_{nu}g_{uv}g_{vm}}{y^{2m+2n+2u+2v} z^{m+n+u+v}}. \tag{12}$$

The perimeter generating function $G(y)$ is obtained by setting $z = 1$ in (12). Using (9), the summations in (12) are straightforward to carry out, with the result that

$$G(y) = \frac{16y^{12}}{(1-4y^2)^2(1-\sqrt{1-4y^2})^4}. \tag{13}$$

The first few terms in the expansion of (13) are $G(y) = y^4+4y^6+18y^8+80y^{10}+351y^{12}+\mathrm{O}(y^{14})$, which agrees with direct enumeration of the lowest-order clusters. For convenience, and for comparison with later results, the basic clusters (excluding rotations) of perimeter length $\ell \leqslant 10$ are shown in figure 5. Clusters are defined with respect to a given origin; thus, for example, the three basic six-site clusters depicted in figure 5 are distinct even though they are translations of each other.

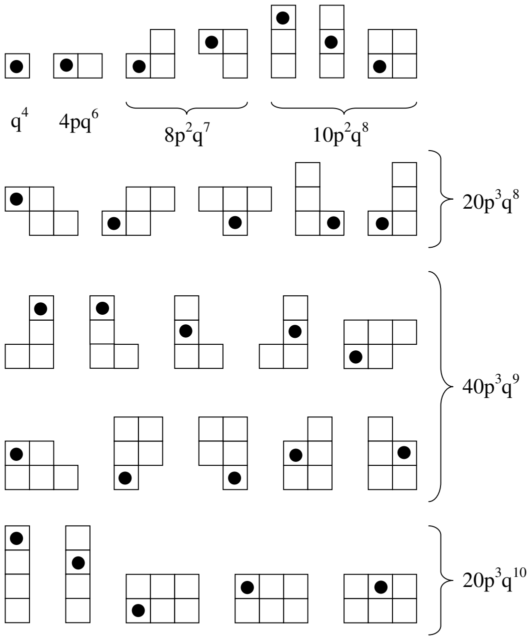

Figure 5. Lowest-order cluster diagrams with perimeter length $\ell \leqslant 10$ (excluding rotations) and their associated weights (including rotations). The full circles indicate the origin site for each cluster. The diagrams are complete to $\mathrm{O}(p^3)$ and $\mathrm{O}(q^8)$ for p, q small respectively.

4. Evaluation of various moments

The probabilistic weighting of CCG clusters is more complicated than for CDP (see the discussion in section 2), but can be handled as follows. We consider first the probability of generating a finite cluster, Q. To begin, one defines a new variable y' through the relation $y \equiv \sqrt{pq}\,y'$. Next, one makes this substitution in (12) and sets $z = 1$. On its own this is not sufficient to get the weighting of each cluster correct. An additional factor of the form $p^{-2}q^{m+n+u+v-2}$ must be introduced into the summand of (12) (i.e. before the summations are carried out) to properly account for the weighting of the common axis sites (recall the discussion in the previous section). Again, proof that this procedure is correct comes from examining a term-by-term diagrammatic expansion. Thus we obtain from (12) and (9) a new generating function:

$$\Pi(p, q, y') = p^4 q^4 y'^{12} \sum_{m,n,u,v=1}^{\infty} \frac{\lambda_-^{2m+2n+2u+2v-8}}{y'^{2m+2n+2u+2v}\, p^{m+n+u+v}}.$$

The various geometric series are easy to evaluate, the result being

$$\Pi(p, q, y') = \left(\frac{pqy'^3}{y'^2 p - \lambda_-^2}\right)^4. \tag{14}$$

The significance of this generating function is as follows. Its expansion as a power series in y' enumerates the probability of generating clusters of perimeter ℓ. The first few terms are

$$\begin{aligned}\Pi(p, q, y') = {} & q^4 y'^4 + 4pq^6 y'^6 + 8p^2q^7 y'^8 + 10p^2q^8 y'^8 \\ & + 20p^3q^8 y'^{10} + 40p^3q^9 y'^{10} + 20p^3q^{10} y'^{10} + \cdots\end{aligned}$$

and these correspond exactly to the diagrams depicted in figure 5. A consequence of (14) is that the weight of a cluster of perimeter ℓ is of the form $p^{-2}p^{\ell/2}q^{\alpha}$, where α is such that $\ell/2+3 \leqslant \alpha \leqslant \ell$ for $\ell > 4$ and all values occur. Thus (as stated in section 2) the perimeter length alone is not sufficient to determine the weighting. We return to this point in the discussion below.

Setting $y' = 1$ in (14) provides an expression for Q. For $p < p_c$ we have $\lambda_- = p$ and $Q_< = 1$. For $p > p_c$ we have $\lambda_- = q$, whereupon

$$Q_> = \left(\frac{pq}{p-q^2}\right)^4. \tag{15}$$

The probability of generating an infinite cluster $P_\infty \equiv 1 - Q$. Slightly above p_c we have $P_\infty \sim 32(p - p_c)$ which has the same critical exponent $\beta = 1$ as CDP.

From (14) one can also calculate the mean cluster perimeter length. For $p < p_c$,

$$L_< = \left.\frac{\partial \Pi}{\partial y'}\right|_{y'=1} = 4 + \frac{8p}{(1-p)(1-2p)}. \tag{16}$$

The divergence is governed by the CDP critical exponent $\tau = 1$. The mean cluster perimeter length for $p > p_c$ (given that the clusters are finite) is

$$L_> = Q_>^{-1} \left.\frac{\partial \Pi}{\partial y'}\right|_{y'=1} = 4 + \frac{8q^2}{(1-q-q^2)(1-2q)}. \tag{17}$$

It is clear from (16) and (17) that the CDP duality property under the transformation $p \Leftrightarrow q$ no longer holds, although the divergence $\sim 4|p - p_c|^{-1}$ is the same on both sides of the transition. This 'lack of duality' is confirmed by series expansions; for small p we have $L_< = 4 + 8p + 24p^2 + 56p^3 + \mathrm{O}(p^4)$, whilst for small q we have $L_> = 4 + 8q^2 + 24q^3 + 64q^4 + \mathrm{O}(q^5)$, both of which may be verified from the diagrams in figure 5 (weighted by Q^{-1}). We comment further on this below.

The remaining quantity of particular interest is the mean cluster size, S. To evaluate it, we first introduce the factor $p^{-2}q^{m+n+u+v-2}$ into the summand of (12). Next we differentiate (12) with respect to z, before finally setting $z = 1$ and $y = \sqrt{pq}$. Proof that this procedure is correct also comes from examining a term-by-term expansion. The result (after exploiting various symmetries), which is valid on either side of the transition (given that the clusters are finite), is

$$S = 1 + \frac{4Q^{-1}}{p^4q^4} \sum_{m,n,u,v=1}^{\infty} \left.\frac{(f_{mn} - mg_{mn})g_{nu}g_{uv}g_{vm}}{p^{m+n+u+v}}\right|_{\substack{y=\sqrt{pq}\\ z=1}}.$$

The function f_{mn} is defined in (11). The various summations are straightforward to carry out, although rather laborious. For $p < p_c$ one finds that

$$S_< = 1 + \frac{4p(1-p-p^2)}{(1-p)(1-2p)^2}. \tag{18}$$

Thus the divergence near p_c is governed by the CDP exponent $\gamma = 2$. For $p > p_c$, the mean cluster size (given that the clusters are finite) is

$$\begin{aligned} S_> &= 1 + \frac{4q^2K(q)}{(1-q-q^2)^2(1-2q)^2} \\ K(q) &= (1 - 3q + 2q^2 + 2q^3 - 3q^4). \end{aligned} \tag{19}$$

There does not appear to be an 'elegant' simplification of $K(q)$. Clearly the duality structure between (18) and (19) is missing, as it was for L, although near p_c the divergence $\sim(2|p_c - p|)^{-2}$

is the same on both sides of the transition. This lack of duality is evident in the series expansions; for small p we have $S_< = 1 + 4p + 16p^2 + 44p^3 + O(p^4)$, whilst for small q we have $S_> = 1+4q^2+12q^3+36q^4+O(q^5)$, both of which may be verified from the diagrams in figure 5 (weighted by Q^{-1}).

5. Discussion

That the CCG model can be solved exactly is due to its close link with CDP. An immediate consequence is that the critical exponents of the two models are the same. Further, for large clusters the four 'lobes' decouple and one would expect each lobe to begin to resemble more and more an independent CDP cluster as growth continues. One can see this in the expressions for P_∞, L and S. Near p_c they are all a factor of four larger in the present model than in CDP.

The principal difference between the present model and CDP in (1 + 1) dimensions is that the cluster boundary is less restricted. This is evident in the fact that the perimeter length alone no longer determines the cluster weightings. The perimeter polygons defined by the model are self-avoiding and distinct from other widely studied cases, such as staircase polygons, partition polygons and convex polygons [7]. The Eden model also generates compact clusters with non-trivial boundaries [25], although for that model compactness is meant in the more general sense that the cluster's fractal dimension $d_f = 2$. In the present problem it is trivial that $d_f = 2$. As an observation one generally expects the cluster radius of gyration $R_N \sim N^\nu$, where N is the cluster size and the exponent $\nu = 1/d_f$ [25]. In the present problem, if one considers large clusters near p_c, $N \sim S \sim |p - p_c|^{-2}$ and $R_N \sim L \sim |p - p_c|^{-1}$, so that $R_N \sim N^{1/2}$ and the scaling relationship is satisfied.

The fact that the perimeter length alone is not sufficient to determine the cluster weightings also explains the absence of the duality feature observed in CDP in (1 + 1) dimensions under the transformation $p \Leftrightarrow q$. The latter relies on the fact that, in every cluster's weighting, a factor of q is accompanied by a factor of p. Once this feature breaks down there is no longer any reason to expect the duality symmetry to hold exactly, although near p_c it is approximately preserved.

We have shown how to calculate various low order moments by considering area–perimeter generating functions and their derivatives. One could, in principle, extend the calculations to include higher-order moments, but the effort involved would be considerable. There are, however, several other avenues that may be worth pursuing. As a technical exercise, it should be possible to solve (8) for all values of z and explicitly write down the area–perimeter generating function (12) in terms of 'q series'. Rather more interesting would be to see whether the present results could be recast in terms of simple non-linear functional equations for the generating functions of interest (especially (12)) [10]. Finally, it is not inconceivable that the present model might be amenable to exact treatment as regards some of its dynamic (growth) properties [5], although this remains to be established.

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